

SEMISTABILITY OF SYZYGY BUNDLES ON PROJECTIVE SPACES IN POSITIVE CHARACTERISTICS

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1. INTRODUCTION

Let k be an algebraically closed field. For an integer $d > 0$, let \mathcal{V}_d be the vector bundle on \mathbf{P}_k^n given by the exact sequence

$$(1.1) \quad 0 \longrightarrow \mathcal{V}_d \longrightarrow H^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(d)) \otimes \mathcal{O}_{\mathbf{P}_k^n} \xrightarrow{\eta} \mathcal{O}_{\mathbf{P}_k^n}(d) \longrightarrow 0,$$

where η is the evaluation map.

It was proved by Flenner [F] that if characteristic $k = 0$ then \mathcal{V}_d is a semistable vector bundle. He uses this as an crucial ingredient to prove his restriction theorem for torsion free semistable sheaves on a normal projective variety, defined over a field of characteristic 0, to a general hypersurface of degree d , where d has a lower bound in terms of degree of the ambient variety and degree and rank of the sheaf. He reduces the argument to projective space and then uses the semistability property of \mathcal{V}_d .

In characteristic $k = p > 0$, A. Langer ([L], proved the following restriction theorem for strongly semistability:

Theorem 1.1. *(Langer) Let (X, H) be a smooth n -dimensional ($n \geq 2$) polarized variety with globally generated tangent bundle \mathcal{T}_X . Let E be a H -semistable torsion free sheaf of rank $r \geq 2$ on X . Let d be an integer such that*

$$d > \frac{r-1}{r} \Delta(E) H^{n-2} + \frac{1}{r(r-1)H^n}$$

and

$$\frac{\binom{d+n}{d} - 1}{d} > H^n \max\left\{\frac{r^2 - 1}{4}, 1\right\} + 1.$$

If characteristic $k > d$ then the restriction E_D is strongly H -semistable for a very general $D \in |dH|$.

However, he has to assume that characteristic $k = p > d$; as he uses the proof of the result of Flenner, and more specifically the semistability property of \mathcal{V}_d . In particular, for a given $p > 0$ his result is valid for at most finitely many d , in fact the set of such d can be empty. In the end of the proof of Theorem 1.1, Langer remarked that the assumption on the characteristic could be removed if there is a positive answer to one of the following questions:

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preliminary version.

Is \mathcal{V}_d a semistable bundle, for arbitrary n , d , and $p = \text{char } k$?, or is there a good estimate on $\mu_{\max}(\mathcal{V}_d^*)$?

We recall that if $\text{char } k = p > d$ or $\text{char } k = 0$, then \mathcal{V}_d is filtered by $S^m(\mathcal{V}_1) \otimes \mathcal{O}(d-m)$, and these are the only possible subquotients of \mathcal{V}_d as homogeneous bundles. However, as soon as d exceeds p , many more subquotients of \mathcal{V}_d occur, and therefore argument of [F] is not applicable.

In this paper, we prove semistability of the syzygy bundle \mathcal{V}_d , where $\text{char } k = p > 0$, under the conditions as given in Theorems 1.2. This provides evidence in favour of a positive solution in general.

Theorem 1.2. *The vector bundle \mathcal{V}_d , given by the short exact sequence (1.1), is semistable (in fact stable) in any of the following cases:*

- (1) $\mathbf{P}_k^n = \mathbf{P}_k^2$ and $d \geq 1$, or
- (2) $d = a_i p^i + a_m p^m$ is the p -adic expansion of the integer d , for any $m \geq 0$, or
- (3) $d = (a_0 + a_1 p + \cdots + a_m p^m) p^{i_0}$ is the p -adic expansion of the integer d , where a_0 and a_m are nonzero integers, such that one of the following holds,
 - (a) $p \leq n$ and $a_2, \dots, a_m \geq 1$, or
 - (b) $p \geq n$ and $a_2, \dots, a_m \geq p - n + 1$, or
 - (c) $n \geq (a_0 + a_1 p + \cdots + a_m p^m)/p$.

By analysing further the proof of Theorem 1.2, we answer the second question of Langer affirmatively in the following

Proposition 1.3. *Let \mathcal{V}_d be the vector bundle on \mathbf{P}_k^n , given by the short exact sequence (1.1). Let $\mathcal{V}_d^* = \mathcal{H}om_{\mathcal{O}_{\mathbf{P}_k^n}}(\mathcal{V}_d, \mathcal{O}_{\mathbf{P}_k^n})$ denote the dual of \mathcal{V}_d . Then*

$$\frac{d}{\binom{d+n}{d} - 1} \leq \mu_{\max}(\mathcal{V}_d^*) \leq \frac{d}{\binom{\lceil d/2 \rceil + n - 1}{\lceil d/2 \rceil}},$$

where $\lceil x \rceil =$ the smallest integer $\geq x$.

As a consequence one can remove the restriction on the characteristic of the field in Theorem 1.1 of Langer and obtain the following

Corollary 1.4. *Let (X, H) be a smooth n -dimensional ($n \geq 2$) polarized variety with globally generated tangent bundle \mathcal{T}_X . Let E be an H -semistable torsion free sheaf of rank $r \geq 2$ on X . Let d be an integer such that*

$$d > \frac{r-1}{r} \Delta(E) H^{n-2} + \frac{1}{r(r-1)H^n}$$

and,

- (1) for $n = 2$,

$$\frac{\binom{d+n}{d} - 1}{d} > H^n \max\left\{\frac{r^2 - 1}{4}, 1\right\} + 1,$$

- (2) and, for $n \geq 3$,

$$\frac{\binom{\lceil d/2 \rceil + n - 1}{\lceil d/2 \rceil}}{d} > H^n \max\left\{\frac{r^2 - 1}{4}, 1\right\} + 1.$$

Then the restriction E_D is strongly H -semistable for a very general $D \in |dH|$.

It follows from the above corollary that, for a given char $k = p > 0$, one can find d_0 such that, for all $d \geq d_0$, the restriction E_D is strongly H -semistable for a very general $D \in |dH|$.

We recall that (1) when X is a smooth projective variety with a polarization H and E a strongly H -semistable bundle on X of rank $< \dim X$ then Maruyama [Ma] proved that $E|_D$ is strongly H -semistable for a very general $D \in |H|$, i.e., $d = 1$ in this case, (2) when E is a homogeneous bundle on \mathbf{P}_k^n induced by an irreducible representation of P , where P is a maximal parabolic subgroup of $GL(n+1)$ such that $GL(n+1)/P = \mathbf{P}_k^n$, then $E|_D$ is strongly semistable, (i) if D is any smooth quadric and char $k \neq 2$ and (ii) if D is any smooth cubic and char $k \neq 3$. In particular $E|_D$ is strongly semistable for very general hypersurface of degree ≥ 2 , if char $k > 5$.

Remark 1.5. One can prove the semistability of \mathcal{V}_d in the following case also, but the proof gets very technical (see arXiv:mathRT/0804.0547): Let \mathcal{V}_d be the vector bundle as given by the short exact sequence (1.1). Suppose $d = (a_0 + a_1p + \cdots + a_mp^m)p^{i_0}$ is the p -adic expansion of the integer d , where a_0 and a_m are nonzero integers, such that

- (1) $n \geq m + 1$,
- (2) $h^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(a_m)) \geq 1 + a_mmn$, and
- (3) $a_0 \leq a_1 \leq \cdots \leq a_{m-1}$ and $a_{m-2} \leq a_m$.

Then \mathcal{V}_d is a semistable (in fact stable) vector bundle over \mathbf{P}_k^n .

If $a_m \geq 4$ then the condition (2) in the above statement is always satisfied. Moreover if $m \leq n - 2$ then the condition (2) is satisfied for any $a_m \geq 3$.

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2. SYZGY BUNDLES ON \mathbf{P}_k^2

First we prove the following result for \mathbf{P}_k^2 , by different methods than we use for higher dimensional projective space.

Proposition 2.1. *For $n = 2$ and for $d \geq 1$, the bundle \mathcal{V}_d is stable, on \mathbf{P}_k^2 .*

The proof relies on the following lemma, which we prove using an argument similar to the proof the following proposition in [KR].

Proposition [KR] *Let X be a nonsingular curve of genus $g \geq 2$. Then for the pair (X, ω_X) , where ω_X is the canonical line bundle of X , the sheaf K_{ω_X} is semistable.*

Lemma 2.2. *Let X be a nonsingular curve of genus $g \geq 2$ and \mathcal{L} be a (base point free) line bundle on X such that $\deg \mathcal{L} > 2g$. Let $K_{\mathcal{L}}$ be the syzygy bundle for the evaluation map $H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \rightarrow \mathcal{L}$. Then $K_{\mathcal{L}}$ is stable.*

Proof. Consider the short exact sequence

$$0 \longrightarrow K_{\mathcal{L}} \longrightarrow H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow 0.$$

Now $h^0(X, \mathcal{L}) = \deg \mathcal{L} + 1 - g$, since $H^1(X, \mathcal{L}) = 0$, by Serre duality. Therefore

$$\text{rank } K_{\mathcal{L}} = \deg \mathcal{L} - g \text{ and } \det K_{\mathcal{L}}^{\vee} = \mathcal{L},$$

so that $\deg K_{\mathcal{L}}^{\vee} = \deg \mathcal{L}$. This implies

$$\text{slope } (K_{\mathcal{L}}^{\vee}) = \deg \mathcal{L} / (\deg \mathcal{L} - g) < 2.$$

Let \mathcal{F} be a quotient bundle of $K_{\mathcal{L}}^{\vee}$; then \mathcal{F} is generated by its global sections, and $h^0(X, \mathcal{F}) \geq r + 1$ if $\text{rank } \mathcal{F} = r$ (otherwise it would contradict the fact that $h^0(X, K_{\mathcal{L}}) = 0$, because if \mathcal{F} is trivial, then so is \mathcal{F}^{\vee} , and this would imply that $h^0(X, K_{\mathcal{L}}) \geq r$). We choose (see [KR]) $W \subseteq H^0(X, K_{\mathcal{L}}^{\vee})$ such that $\dim W = r + 1$ and W generates \mathcal{F} ; let

$$0 \longrightarrow \mathcal{M} \longrightarrow W \otimes \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0$$

be the corresponding short exact sequence. Then \mathcal{M} is a line bundle, isomorphic to $\wedge^r \mathcal{F}^{\vee}$, so that $\deg \mathcal{F} = \deg \mathcal{M}^{\vee}$. Note that $H^0(X, \mathcal{F}^{\vee}) = 0$ as $H^0(X, K_{\mathcal{L}}) = 0$ and therefore $\dim H^0(X, \mathcal{M}^{\vee}) \geq r + 1$.

- (1) Suppose $H^1(X, \mathcal{M}^{\vee}) \neq 0$; then by Clifford's theorem (chap IV, Theorem 5.4 [H])

$$\dim H^0(X, \mathcal{M}^{\vee}) - 1 \leq (1/2) (\deg \mathcal{M}^{\vee}).$$

Hence $r \leq (\deg \mathcal{F})/2$. This implies that $\text{slope } \mathcal{F} \geq 2 > \text{slope } K_{\mathcal{L}}^{\vee}$.

- (2) Suppose $H^1(X, \mathcal{M}^{\vee}) = 0$. Then

$$H^0(X, \mathcal{M}^{\vee}) = \deg \mathcal{M}^{\vee} + 1 - g.$$

Hence $r + 1 \leq \deg \mathcal{F} + 1 - g$. This implies

$$\text{slope } \mathcal{F} \geq 1 + g/r > 1 + g/(\deg \mathcal{L} - g) = \deg \mathcal{L} / (\deg \mathcal{L} - g) = \text{slope } K_{\mathcal{L}}^{\vee}.$$

This proves the lemma. \square

Proof of Proposition 2.1: Choose a nonsingular curve X of degree d in \mathbf{P}_k^2 . We can deduce that

$$H^0(\mathbf{P}_k^2, \mathcal{O}_{\mathbf{P}_k^2}(d)) \simeq H^0(X, \mathcal{O}_X(d))$$

from the following short exact sequence of sheaves of $\mathcal{O}_{\mathbf{P}_k^2}$ -modules

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}_k^2}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}_k^2}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

Let $\mathcal{L} = \mathcal{O}_{\mathbf{P}_k^2}(d) |_{X} = \mathcal{O}_X(d)$. Then we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{V}_d |_X & \longrightarrow & H^0(\mathbf{P}_k^2, \mathcal{O}_{\mathbf{P}_k^2}(d)) \otimes \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\mathbf{P}_k^2}(d) |_X \longrightarrow 0 \\ & & & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & K_{\mathcal{L}} & \longrightarrow & H^0(X, \mathcal{L}) \otimes \mathcal{O}_X & \longrightarrow & \mathcal{L} \longrightarrow 0. \end{array}$$

This implies that $\mathcal{V}_d |_X \cong K_{\mathcal{L}}$. But $\deg \mathcal{L} = d^2 > 2(\text{genus } X)$. Therefore, by Lemma 2.2, the bundle $K_{\mathcal{L}}$ is stable on X . Hence the bundle \mathcal{V}_d is stable on \mathbf{P}_k^2 . This proves the proposition. \square

3. THE HIGHER DIMENSIONAL CASE

Henceforth we assume that $n \geq 3$. Let $G = GL_{n+1}(k)$ and let P be the maximal parabolic group of G given by

$$P = \left\{ \begin{bmatrix} g_{11} & * \\ 0 & A \end{bmatrix} \in GL(n+1), \text{ where } A \in GL(n) \right\}.$$

Then there exists a canonical isomorphism $G/P \simeq \mathbf{P}_k^n$. Recall that there is an equivalence of categories between homogeneous G -bundles on G/P and (finite dimensional) P -modules. The short exact sequence

$$0 \longrightarrow \mathcal{V}_d \longrightarrow H^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(d)) \otimes \mathcal{O}_{\mathbf{P}_k^n} \longrightarrow \mathcal{O}_{\mathbf{P}_k^n}(d) \longrightarrow 0,$$

is naturally a sequence of homogeneous G -bundles, which corresponds to the short exact sequence of P -modules,

$$0 \longrightarrow V_d \longrightarrow U_d \longrightarrow W_d \longrightarrow 0,$$

given as follows. Let U_1 and V_1 be k -vector spaces given by the basis $\{x_1, \dots, x_n, z\}$ and $\{x_1, \dots, x_n\}$ respectively. Then U_1 is a P -module such that if

$$g = \begin{bmatrix} g_{11} & * \\ 0 & A \end{bmatrix}, \text{ where } A \in GL(n),$$

is an element of P then the representation $\rho : P \longrightarrow GL(U_1)$ is defined as follows (by matrix multiplication)

$$\rho(g)(z, x_1, \dots, x_n) = [z, x_1, \dots, x_n] \cdot g^{-1}.$$

This gives canonical action of P on $U_d = S^d(U_1)$ and on

$$V_d = (S^1(V_1) \otimes z^{d-1}) \oplus (S^2(V_1) \otimes z^{d-2}) \oplus \dots \oplus S^d(V_1),$$

where elements of U_d are homogeneous polynomials of degree d in x_1, \dots, x_n, z .

Lemma 3.1. *Let $\mathcal{W} \subseteq \mathcal{V}_d$ be a homogeneous G -bundle and let $W \subset V_d$ be the corresponding P -module.*

- (1) *Suppose $f_0 + f_1 z + \dots + f_m z^m \in W$, where $f_i \in S^{d-i}(V_1)$. Then $f_i z^i \in W$, for all $i \geq 0$. In other words, as k -vector spaces*

$$W = W \cap (S^1(V_1) \otimes z^{d-1}) \oplus W \cap (S^2(V_1) \otimes z^{d-2}) \oplus \dots \oplus W \cap S^d(V_1).$$

Moreover,

- (2) *if $i = i_0 + i_1 p + \dots + i_m p^m$ denotes the p -adic expansion of a positive integer i and if $f \in S^{t_0}(V_1)$ is a homogeneous polynomial such that $(f)z^i \in W$ (in particular $t_0 + i = d$) then*

$$\begin{aligned} W \supseteq & \phi[f \otimes \{S^{i_0}(V_1) \oplus (S^{i_0-1}(V_1) \otimes z) \oplus \dots \oplus z^{i_0}\} \otimes \\ & F^* \{S^{i_1}(V_1) \oplus (S^{i_1-1}(V_1) \otimes z) \oplus \dots \oplus z^{i_1}\} \otimes \dots \\ & \otimes F^{m*} \{S^{i_m}(V_1) \oplus (S^{i_m-1}(V_1) \otimes z) \oplus \dots \oplus z^{i_m}\}], \end{aligned}$$

where, for a positive integer j such that $j = j_0 + j_p + \cdots + k_m p^m$, with the condition that $0 \leq j_k \leq i_k$, the map

$$\phi : S^{t_0}(V_1) \otimes_{j=0}^m F^{j*} \left[\bigoplus_{k_j=0}^{i_j} (S^{i_j-k_j}(V_1) \otimes z^{k_j}) \right] \rightarrow \bigoplus_{j=0}^m \bigoplus_{k_j=0}^{i_j} (S^{t_0+(i_j-k_j)p^j}(V_1) \otimes z^{i-(i_j-k_j)p^j})$$

is the canonical map mapping to V_d , and F^t denote the t^{th} -iterated Frobenius morphism and, for a vector-space U generated by $\{u_1, \dots, u_t\}$, the vector-space $F^{i*}(U)$ is generated by $\{u_1^{p^i}, \dots, u_t^{p^i}\}$.

Proof. The first part follows from the fact that the diagonal Torus group $T \subseteq GL(n+1)$ is contained in P .

To prove the second part of the lemma, by looking at possible monomials occuring on the right side of \supseteq , we see that it is enough to prove the following: Let $k < i$ be a nonnegative integer so that if we have $k = k_0 + k_1 p + \cdots + k_m p^m$ with $0 \leq k_j \leq i_j$, for $0 \leq j \leq m$. Let $x_1^{T_1} \cdots x_n^{T_n} \in S^{i-k}(V_1)$ be a monomial with

$$T_j = t_{0j} + t_{1j}p + \cdots + t_{mj}p^j, \text{ where, } 0 \leq t_{ij} \leq p-1$$

such that

$$(t_{01} + \cdots + t_{0n}) + (t_{11} + \cdots + t_{1n})p + \cdots + (t_{m1} + \cdots + t_{mn})p^m = i - k,$$

where $t_{j1} + \cdots + t_{jn} = i_j - k_j$. Then $(f)(x_1^{T_1} \cdots x_n^{T_n})z^k \in W$.

As W is a P -module, $(f)z^i \in W$ implies that $(f)(bz + a_1x_1 + \cdots + a_nx_n)^i \in W$, for every $(b, a_1, \dots, a_n) \in (k \setminus \{0\}) \times k^n$. Let $y = a_1x_1 + \cdots + a_nx_n$. Now $(f)(bz + y)^i \in W$ implies that

$$(f) \left[\binom{i}{1} (bz)^{i-1}y + \cdots + \binom{i}{i-1} (bz)y^{i-1} + y^i \right] \in W.$$

Hence, as argued in part (1) of the lemma, we have $(f)\binom{i}{k}y^{i-k}z^k \in W$, for every $0 \leq k \leq i$. Now

$$\binom{i}{k} = \frac{(i-k+k) \cdots (i-k+1)}{k(k-1) \cdots 1},$$

where the terms, divisible by p , in the numerator are

$$\{(i-k-(i_0-k_0)+lp) \mid 1 \leq l \leq k_1 + \cdots + k_m p^{m-1}\}$$

and in the denominator are

$$\{lp \mid 1 \leq l \leq k_1 + \cdots + k_m p^{m-1}\}.$$

Hence if $k_j \leq i_j$, for all $0 \leq j \leq m$, then $\text{g.c.d.}(\binom{i}{k}, p) = 1$ which implies $(f)y^{i-k}z^k \in W$. That is, $(f)(a_1x_1 + \cdots + a_nx_n)^{i-k}z^k \in W$, for every $(a_1, \dots, a_n) \in k^n$.

Now let $y_1 = a_2x_2 + \cdots + a_nx_n$, so that $(f)(y_1 + a_1x_1)^{i-k}z^k \in W$. Since $T_1 \leq i-k$ and $t_{j1} \leq i_j - k_j$, where

$$t_{01} + t_{11}p + \cdots + t_{m1}p^m = T_1 \text{ and } (i_0 - k_0) + (i_1 - k_1)p + \cdots + (i_m - k_m)p^m = i - k,$$

such that $0 \leq t_{ij}, i_j - k_j \leq p-1$. Hence, by a similar argument with a binomial expansion, we have $(f)y_1^{i-k-T_1}x_1^{T_1}z^k \in W$. Iterating the argument we deduce

that $(f)x_1^{T_1} \cdots x_n^{T_n} \in W$. This completes the proof of the claim (and hence of the lemma). \square

Now throughout this paper, we fix a positive integer d with its p -adic expansion $d = a_0 + a_1p + \cdots + a_mp^m$ and we also fix a nonzero homogeneous G -subbundle $\mathcal{W} \subseteq \mathcal{V}_d$ given by the corresponding P -module $W \subset V_d$. We would denote

$$W(i) = W \cap (S^{d-i}(V_1) \otimes z^i).$$

By Lemma 3.1, $W = \bigoplus_{i=0}^{d-1} W(i)$, such that \mathcal{W} has a filtration by G -subbundles $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_{d-1} = \mathcal{W}$, where \mathcal{F}_i is the vector bundle associated to the P -submodule $W(0) \oplus W(1) \oplus \cdots \oplus W(i) \subseteq W$. In particular, the vector-subspace $W(i)$ has the *canonical P -subquotient* module structure with associated homogeneous G -bundle $\mathcal{W}(i)$, so that there is a G -equivariant isomorphism

$$\mathrm{gr}(\mathcal{W}) \cong \bigoplus_{i=0}^{d-1} \mathcal{W}(i).$$

For a k -vector space V , the number $|V|$ denotes the dimension of V .

Remark 3.2. Let $i, j < d$ be two positive integers such that $i = i_0 + i_1p + \cdots + i_mp^m$ and $j = j_0 + j_1p + \cdots + j_mp^m$ with the condition that $0 \leq j_k \leq i_k \leq p-1$, for every $k \geq 0$. Then, by part (2) of Lemma 3.1,

$$W(i) \neq 0 \implies W(j) \neq 0.$$

Moreover, if $W(i) = B_i \otimes z^i$, where $B_i \subseteq S^{d-i}(V_1)$, then

$$W(j) \supseteq \phi \left[B_i \otimes S^{i_0-j_0}(V_1) \otimes F^*(S^{i_1-j_1}(V_1)) \otimes \cdots \otimes F^{*m}(S^{i_m-j_m}(V_1)) \right] \otimes z^j,$$

where

$$\phi : S^{d-i}(V_1) \otimes S^{i_0-j_0}(V_1) \otimes F^*(S^{i_1-j_1}(V_1)) \otimes \cdots \otimes F^{*m}(S^{i_m-j_m}(V_1)) \rightarrow S^{d-j}(V_1)$$

is the canonical map.

Lemma 3.3. *If $W(i) \neq 0$ and if $d-i = (t_0+t_1p+\cdots+t_mp^m)$, where $0 \leq t_i \leq p-1$ then*

$$S^{t_0}(V_1) \otimes F^*(S^{t_1}(V_1)) \otimes \cdots \otimes F^{*m}(S^{t_m}(V_1)) \otimes z^i \subseteq W.$$

Proof. $W(i) \neq 0$ means $(S^{d-i}(V_1) \otimes z^i) \cap W \neq 0$. Therefore $W = W_1 \otimes z^i$, for some nonzero $SL(n)$ -submodule W_1 of $S^{d-i}(V_1)$ (with respect to the induced action on $S^{d-i}(V_1)$ coming from the canonical action of $SL(n)$ on V_1). But (see [B]), for the p -adic expansion $t_0+t_1p+\cdots+t_mp^m = d-i$ of the integer $d-i$, the $SL(n)$ -module $S^{t_0}(V_1) \otimes F^*(S^{t_1}(V_1)) \otimes \cdots \otimes F^{*m}(S^{t_m}(V_1))$ is the smallest $SL(n)$ -submodule of $S^{d-i}(V_1)$. In particular, it is contained in W_1 . This proves the lemma. \square

Remark 3.4. For a vector-bundle \mathcal{V} on \mathbf{P}_k^n , with determinant $\det(\mathcal{V}) = \mathcal{O}_{\mathbf{P}_k^n}(m)$, we define $\deg(\mathcal{V}) = m$ and $\mu(\mathcal{V}) = \deg(\mathcal{V})/\mathrm{rank}(\mathcal{V})$. We note that

$$\deg \mathcal{W} = \sum \deg \mathcal{W}(i).$$

Since \mathcal{V}_1 is a semistable vector bundle on \mathbf{P}_k^n , by Theorem 2.1 of [MR], the vector bundle $S^{d-i}(\mathcal{V}_1)$ is semistable on \mathbf{P}_k^n . Hence

$$-\deg \mathcal{W}(i) \geq -\mu(S^{d-i}(\mathcal{V}_1) \otimes \mathcal{O}(i))|W(i)|.$$

Now we prove a series of lemmas before coming to the main result.

Lemma 3.5. *If $d = a_0 < p$ then, for $0 \subsetneq \mathcal{W} \subsetneq \mathcal{V}_d$, we have $-\deg \mathcal{W} \geq a_0$. In particular $\mu(\mathcal{W}) < \mu(\mathcal{V}_d)$.*

Proof. For $d = a_0 \leq p - 1$, the P -module V_{a_0} is filtered by the subquotients isomorphic to $S^{d-i}(V_1) \otimes z^i$, where $0 \leq i < a_0$. If i_0 is the largest integer with the property that $W(i_0) \neq 0$ then, by Remark 3.2 and Lemma 3.3, it follows that $W = \sum_{j=0}^{i_0} S^{d-j}(V_1) \otimes z^j$, as P -module. Therefore

$$\begin{aligned} -\deg \mathcal{W} &= \sum_{j=0}^{i_0} -\deg (S^{d-j}(\mathcal{V}_1) \otimes z^j) \\ &= -\sum_{j=0}^{i_0} \mu(S^{d-j}(\mathcal{V}_1) \otimes z^j) |S^{d-j}(V_1)| \\ &= \frac{1}{n} \sum_{j=0}^{i_0} [(d-j) - nj] |S^{d-j}(V_1)| \\ &= (i_0 + 1) |S^{a_0-i_0-1}(V_1)| \geq a_0, \end{aligned}$$

where the second last equality follows from the fact that, for an integer $a \geq 0$, we have

$$(a+1)|S^{a+1}(V_1)| = n(|S^a(V_1)| + \cdots + |S^1(V_1)| + |S^0(V_1)|).$$

This proves the lemma. \square

In the rest of this section, we assume that the integer d has the p -adic expansion $d = a_0 + a_1p + \cdots + a_mp^m$ such that a_0 and a_m are nonzero integers.

Remark 3.6. For $i_0 + \cdots + i_mp^m < a_0 + \cdots + a_mp^m$ (where $0 \leq i_0, \dots, i_m \leq p-1$), let

$$W(i_0 + i_1p + \cdots + i_mp^m) = W_{i_0, \dots, i_m} = W \cap [S^{d-(i_0 + \cdots + i_mp^m)}(V_1) \otimes z^{i_0 + \cdots + i_mp^m}]$$

be the subspace with canonical P -(subquotient) structure and let $\mathcal{W}_{i_0, \dots, i_m}$ be the associated G -bundle. Then, by Lemma 3.1,

$$\text{gr } \mathcal{W} = \bigoplus_{(i_0, \dots, i_m) \in C_0(\mathcal{W}) \cup \cdots \cup C_m(\mathcal{W})} \mathcal{W}_{i_0, \dots, i_m},$$

where

$$\begin{aligned} C_0(\mathcal{W}) &= \{(i_0, a_1, \dots, a_m) \mid 0 \leq i_0 < a_0 \text{ and } W_{i_0, \dots, a_m} \neq 0\} \\ &\vdots \\ C_j(\mathcal{W}) &= \{(i_0, \dots, i_j, a_{j+1}, \dots, a_m) \mid 0 \leq i_j < a_j \text{ and } W_{i_0, \dots, a_m} \neq 0\} \\ &\vdots \\ C_{m-1}(\mathcal{W}) &= \{(i_0, \dots, i_{m-1}, a_m) \mid 0 \leq i_{m-1} < a_{m-1}, \text{ and } W_{i_0, \dots, a_m} \neq 0\} \\ C_m(\mathcal{W}) &= \{(i_0, \dots, i_{m-1}, i_m) \mid 0 \leq i_m < a_m, \text{ and } W_{i_0, \dots, i_m} \neq 0\} \end{aligned}$$

Note that if $a_j = 0$, for some j then $C_j(\mathcal{W}) = \phi$. Now

$$\begin{aligned} -\mu(\mathcal{V}_d)|W| &= \frac{d}{|V_d|} \left[\sum_{\{(i_0, \dots, i_m) \in C_m(\mathcal{W})\}} |\mathcal{W}_{i_0, \dots, i_m}| + \sum_{\{(i_0, \dots, i_m) \in C_0(\mathcal{W}) \cup \dots \cup C_{m-1}(\mathcal{W})\}} |\mathcal{W}_{i_0, \dots, i_m}| \right] \\ &< (n|C_m(\mathcal{W})|) + (|C_0(\mathcal{W})| + \dots + |C_{m-1}(\mathcal{W})|), \end{aligned}$$

where the last inequality follows, as

(1) for any positive integer $a \geq 1$, we have

$$\frac{a}{|V_a|} |S^a(V_1)| = \frac{a|S^a(V_1)|}{h^0(\mathcal{O}(a)) - 1} = \frac{na|S^a(V_1)|}{(a+1)|S^{a+1}(V_1)| - n} < n.$$

Therefore $\{i_0, \dots, i_m\} \in C_m(\mathcal{W}) \implies \frac{d}{|V_d|} |\mathcal{W}_{i_0, \dots, i_m}| < n$.

(2) the canonical inclusion (but not a surjection)

$$F^{*m} S^{a_m}(V_1) \otimes S^{a_0 + \dots + a_{m-1} p^{m-1}}(V_1) \hookrightarrow S^{a_0 + \dots + a_m p^m}(V_1)$$

implies that $n|S^{a_0 + \dots + a_{m-1} p^{m-1}}(V_1)| < |S^{a_0 + \dots + a_m p^m}(V_1)|$. Now if

$$\{i_0, \dots, i_m\} \in C_0(\mathcal{W}) \cup \dots \cup C_{m-1}(\mathcal{W}), \text{ then } i_m = a_m,$$

which implies that $|W_{i_0, \dots, i_m}| \leq |S^{a_0 + \dots + a_{m-1} p^{m-1}}(V_1)|$. This implies that $\frac{d}{|V_d|} |\mathcal{W}_{i_0, \dots, i_m}| < 1$.

Lemma 3.7. *If $W(a_0) = 0$ then $\mu(\mathcal{W}) < \mu(\mathcal{V}_d)$.*

Proof. Consider the following diagram of G -bundles,

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ 0 \rightarrow \mathcal{V}_{a_0} \otimes F^* \mathcal{V}_{a_1 + \dots + a_m p^{m-1}} & \rightarrow & H(a_0) \otimes F^* \mathcal{V}_{a_1 + \dots + a_m p^{m-1}} & \rightarrow & \mathcal{O}(a_0) \otimes F^* \mathcal{V}_{a_1 + \dots + a_m p^{m-1}} & \rightarrow & 0 \\ & & & \downarrow & & & \\ & & & \mathcal{V}_{a_0 + \dots + a_m p^m} & & & \\ & & & \downarrow & & & \\ 0 \rightarrow \mathcal{V}_{a_0} \otimes \mathcal{O}(a_1 + \dots + a_m p^m) & \rightarrow & \text{coker}(f) & \rightarrow & \oplus \mathcal{O}_X & \rightarrow & 0, \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

where $H(a_0) = H^0(\mathcal{O}(a_0))$ and $f : H(a_0) \otimes F^*(\mathcal{V}_{a_1 + \dots + a_m p^{m-1}}) \rightarrow \mathcal{V}_{a_0 + \dots + a_m p^m}$ is the canonical map. We note that, if \mathcal{V}' denotes the homogeneous bundle $\mathcal{V}' = \text{kernel of the canonical composite map}$

$$\mathcal{V}_{a_0 + \dots + a_m p^m} \rightarrow \text{coker}(f) \rightarrow \oplus \mathcal{O}_X$$

and $V' \subset V_{a_0 + \dots + a_m p^m}$ the corresponding P -submodule, then

$$-\deg \mathcal{W} \geq -\deg \mathcal{W}', \text{ where } W' = W \cap V'.$$

Since $W(a_0) = 0$, we have

$$\mathcal{W}' \subseteq \mathcal{V}_{a_0} \otimes H^0(\mathcal{O}(a_1 + \dots + a_m p^{m-1}))^{(p)},$$

which is a semistable vector bundle over \mathbf{P}_k^n , as \mathcal{V}_{a_0} is semistable (see Lemma 3.5).

Therefore $-\deg \mathcal{W}' \geq \frac{a_0}{|V_{a_0}|} |\mathcal{W}'|$.

We remark that $|C_j(\mathcal{W}')| = |C_j(\mathcal{W})|$, where we define $C_j(\mathcal{W})$ and $C_j(\mathcal{W}')$ as in Remark 3.6: Note $W_{i_0, \dots, i_m} \neq 0$ implies that $i_0 < a_0$ as $W(a_0) = 0$. Let

$$j_0 + \dots + j_m p^m = d - (i_0 + \dots + i_m p^m), \text{ where } 0 \leq j_t \leq p-1,$$

then $j_0 < a_0$. By Lemma 3.3,

$$S^{j_0}(V_1) \otimes \dots \otimes F^{*m} S^{j_m}(V_1) \otimes z^{i_0 + \dots + i_m p^m} \subseteq W_{i_0, \dots, i_m}.$$

Therefore

$$S^{j_0}(V_1) \otimes \dots \otimes F^{*m} S^{j_m}(V_1) \otimes z^{i_0 + \dots + i_m p^m} \subseteq W_{i_0, \dots, i_m} \cap V'_{i_0, \dots, i_m} = W'_{i_0, \dots, i_m}.$$

In particular $W'_{i_0, \dots, i_m} \neq 0$ and $|C_j(\mathcal{W}')| = |C_j(\mathcal{W})|$. Moreover $|W'_{i_0, \dots, i_m}| \geq |S^{j_0}(V_1)| \dots |F^{*m} S^{j_m}(V_1)|$.

But

$$-\deg \mathcal{W}' \geq \frac{a_0}{|V_{a_0}|} \left[\sum_{\{(i_0, \dots, i_m) \in C_m(\mathcal{W}')\}} |W'_{i_0, \dots, i_m}| + \sum_{\{(i_0, \dots, i_m) \in C_0(\mathcal{W}') \cup \dots \cup C_{m-1}(\mathcal{W}')\}} |W'_{i_0, \dots, i_m}| \right]$$

If $(i_0, \dots, i_m) \in C_m(\mathcal{W}')$ then $i_0 < a_0$ and $i_m < a_m$. Therefore, for the p -adic expansion

$$j_0 + \dots + j_m p^m = a_0 + \dots + a_m p^m - (i_0 + \dots + i_m p^m),$$

we have $j_0 > 0$ and $j_1 p + \dots + j_m p^m > 0$. In particular

$$|W'_{i_0, \dots, i_m}| \geq |S^{j_0}(V_1)| \dots |S^{j_m}(V_1)| \geq n |S^{j_0}(V_1)| = n |S^{a_0 - i_0}(V_1)|.$$

Now

$$\begin{aligned} (*) &:= \frac{a_0}{|V_{a_0}|} \sum_{\{(i_0, \dots, i_m) \in C_m(\mathcal{W}')\}} |W'_{i_0, \dots, i_m}| \\ &= \frac{a_0}{|V_{a_0}|} \sum_{\{(i_1, \dots, i_m) | i_m < a_m\}} \sum_{\{k | (k, i_1, \dots, i_m) \in C_m(\mathcal{W}')\}} |W'_{k, i_1, \dots, i_m}| \\ &\geq \frac{a_0}{|V_{a_0}|} \sum_{\{(i_1, \dots, i_m) | i_m < a_m\}} \sum_{\{k | (k, i_1, \dots, i_m) \in C_m(\mathcal{W}')\}} n |S^{a_0 - k}(V_1)|. \end{aligned}$$

Let $I_0(i_1, \dots, i_m) = 0$ if $(k, i_1, \dots, i_m) \notin C_m(\mathcal{W}')$ for all k , otherwise define

$$I_0(i_1, \dots, i_m) = \max \{k_0 + 1 \mid (k_0, i_1, \dots, i_m) \in C_m(\mathcal{W}')\}.$$

Then from the inequality

$$\frac{a_0}{|V_{a_0}|} (|S^{a_0}(V_1)| + \dots + |S^{a_0 - k}(V_1)|) \geq k + 1,$$

for any $0 \leq k \leq a_0 - 1$, it follows that

$$(*) \geq n \sum_{\{(i_1, \dots, i_m) | i_m < a_m\}} I_0(i_1, \dots, i_m) = n |C_m(\mathcal{W}')|.$$

Similarly, we can argue that

$$\begin{aligned} \frac{a_0}{|V_{a_0}|} \sum_{\{(i_0, \dots, i_m) \in C_0(\mathcal{W}') \cup \dots \cup C_{m-1}(\mathcal{W}')\}} |W'_{i_0, \dots, i_m}| &\geq \sum_{\{(i_1, \dots, i_{m-1}, a_m)\}} I_0(i_1, \dots, i_{m-1}, a_m) \\ &= |C_0(\mathcal{W}')| + \dots + |C_{m-1}(\mathcal{W}')|. \end{aligned}$$

This implies

$$\begin{aligned} -\deg \mathcal{W}' &\geq n|C_m(\mathcal{W}')| + (|C_0(\mathcal{W}')| + \cdots + |C_{m-1}(\mathcal{W}')|) \\ &= n|C_m(\mathcal{W})| + (|C_0(\mathcal{W})| + \cdots + |C_{m-1}(\mathcal{W})|). \end{aligned}$$

On the other hand, by Remark 3.6,

$$-\mu(\mathcal{V}_d)|W| < (n|C_m(\mathcal{W})|) + (|C_0(\mathcal{W})| + \cdots + |C_{m-1}(\mathcal{W})|),$$

which implies that $-\mu(\mathcal{V}_d)|W| \leq -\deg \mathcal{W}$. This proves the lemma. \square

Lemma 3.8. *If $W(a_0 + \cdots + a_{m-1}p^{m-1}) \neq 0$ then*

$$-\deg(\mathcal{W}) \geq a_0 + \cdots + a_{m-1}p^{m-1} = -\deg(\mathcal{V}_d).$$

In particular $\mu(\mathcal{W}) < \mu(\mathcal{V}_d)$, if $\mathcal{W} \subsetneq \mathcal{V}$.

Proof. For any $a \in \mathbb{N}$, let $H(a) = H^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(a))$ and let $H(a)^{(p^t)} = F^{*t}(H(a) \otimes \mathcal{O}_{\mathbf{P}_k^n})$, where F^t is the t^{th} iterated Frobenius morphism. Note that there is an exact sequence of G -bundles

$$0 \longrightarrow F^{*t}(\mathcal{V}_a) \longrightarrow H(a)^{(p^t)} \longrightarrow \mathcal{O}_{\mathbf{P}_k^n}(ap^t) \longrightarrow 0.$$

Let δ denote the following tensor product map of G -bundles:

$$\otimes_{i=0}^{m-1} H(a_i)^{(p^i)} \longrightarrow \otimes_{i=0}^{m-1} \mathcal{O}_{\mathbf{P}_k^n}(a_i p^i) = \mathcal{O}_{\mathbf{P}_k^n}(a_0 + a_1 p + \cdots + a_{m-1} p^{m-1}).$$

We then have an induced commutative diagram of homogeneous G -bundles, with exact rows and columns (the term $\oplus \mathcal{O}_{\mathbf{P}_k^n}$ denotes a certain trivial vector bundle, with a G -action),

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ & & & & & & \ker \delta \otimes \mathcal{O}(a_m p^m) \\ & & & & & & \downarrow \\ 0 \rightarrow & \otimes_{i=0}^{m-1} H(a_i)^{(p^i)} \otimes F^{*m} \mathcal{V}_{a_m} \rightarrow & \otimes_{i=0}^{m-1} H(a_i)^{(p^i)} \otimes H(a_m)^{(p^m)} \rightarrow & \otimes_{i=0}^{m-1} H(a_i)^{(p^i)} \otimes \mathcal{O}(a_m p^m) \rightarrow & 0 \\ & \downarrow f & \downarrow & \downarrow & \\ 0 \rightarrow & \mathcal{V}_{a_0 + \cdots + a_m p^m} \rightarrow & H(a_0 + \cdots + a_m p^m) \otimes \mathcal{O}_{\mathbf{P}_k^n} \rightarrow & \mathcal{O}(a_0 + \cdots + a_m p^m) \rightarrow & 0 \\ & \downarrow & \downarrow & \downarrow & \\ & \text{coker}(f) & \oplus \mathcal{O}_{\mathbf{P}_k^n} & 0 & \\ & \downarrow & \downarrow & & \\ & 0 & 0 & & . \end{array}$$

This gives the following diagram of homogeneous G -bundles

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ 0 \rightarrow & \ker \delta \otimes F^{*m} \mathcal{V}_{a_m} \rightarrow & \otimes_{i=0}^{m-1} H(a_i)^{(p^i)} \otimes F^{*m} \mathcal{V}_{a_m} \rightarrow & \mathcal{O}(a_0 + \cdots + a_{m-1} p^{m-1}) \otimes F^{*m} \mathcal{V}_{a_m} \rightarrow & 0 \\ & & \downarrow f & & \\ & & \mathcal{V}_{a_0 + \cdots + a_m p^m} & & \\ & & \downarrow & & \\ 0 \rightarrow & \ker \delta \otimes \mathcal{O}(a_m p^m) \rightarrow & \text{coker}(f) \rightarrow & \oplus \mathcal{O}_{\mathbf{P}_k^n} \rightarrow & 0 \\ & & \downarrow & & \\ & & 0. & & \end{array}$$

We note that, if \mathcal{V}' denotes the homogeneous G -bundle given by $\mathcal{V}' = \text{kernel of the canonical composite map } \mathcal{V}_{a_0+\dots+a_m p^m} \longrightarrow \text{coker}(f) \longrightarrow \bigoplus \mathcal{O}_{\mathbf{P}_k^n}, \text{ i.e.,}$

$$0 \longrightarrow \mathcal{V}' \longrightarrow \mathcal{V}_{a_0+\dots+a_m p^m} \longrightarrow \bigoplus \mathcal{O}_{\mathbf{P}_k^n} \longrightarrow 0,$$

Then

$$-\deg \mathcal{W} \geq -\deg \mathcal{W}', \text{ where } W' = W \cap V',$$

V' is the P -module associated to \mathcal{V}' and \mathcal{W}' is the G -bundle associated to the P -module W' . Let

$$A_{-1} = k.(z^{a_0+\dots+a_{m-1}p^{m-1}})$$

and, for $0 \leq i_0 \leq m-1$, let $A_{i_0} = F^{*i_0} V_{a_{i_0}} \otimes z^{a_0+\dots+\widehat{a_{i_0}p^{i_0}}+\dots+a_{m-1}p^{m-1}}$, where

$$z^{a_0+\dots+\widehat{a_{i_0}p^{i_0}}+\dots+a_{m-1}p^{m-1}} = z^{a_0+\dots+a_{m-1}p^{m-1}-a_{i_0}p^{i_0}}.$$

Inductively, we define

$$A_{i_0\dots i_j} = F^{*i_0} V_{a_{i_0}} \otimes \dots \otimes F^{*i_j} V_{a_{i_j}} \otimes z^{a_0+\dots+\widehat{a_{i_0}p^{i_0}}+\dots+\widehat{a_{i_j}p^{i_j}}+\dots+a_{m-1}p^{m-1}},$$

where $0 \leq i_0 < \dots < i_j \leq m-1$ and

$$z^{a_0+\dots+\widehat{a_{i_0}p^{i_0}}+\dots+\widehat{a_{i_j}p^{i_j}}+\dots+a_{m-1}p^{m-1}} = z^{a_0+\dots+a_{m-1}p^{m-1}-(a_{i_0}p^{i_0}+\dots+a_{i_j}p^{i_j})}.$$

We can write

$$V' = \bigoplus_{j=-1}^{m-1} \bigoplus_{i_0,\dots,i_j} A_{i_0,\dots,i_j} \otimes F^{m*} V_{a_m} + \bigoplus_{j=0}^{m-1} \bigoplus_{i_0,\dots,i_j} A_{i_0,\dots,i_j} \otimes z^{a_m p^m},$$

where by

$$\bigoplus_{i_0,\dots,i_j} \text{ we mean } \bigoplus_{\{0 \leq i_0 < \dots < i_j \leq m-1 \mid A_{i_0,\dots,i_j} \neq 0\}}.$$

Note that $A_{i_0\dots i_j} \neq 0$ if and only if $a_{i_k} \neq 0$, for every $i_k \in \{i_0, \dots, i_j\}$. We also note that $A_{i_0,\dots,i_j} \otimes F^{m*} V_{a_m}$ and $A_{i_0,\dots,i_j} \otimes z^{a_m p^m}$ have canonical P -subquotient module structure. By Lemma 3.1,

$$W' = \bigoplus_{-1 \leq j \leq m-1} \tilde{B}_j \oplus \bigoplus_{0 \leq j \leq m-1} \tilde{C}_j,$$

where

$$\tilde{B}_j = \bigoplus_{i_0,\dots,i_j} B_{i_0,\dots,i_j}, \text{ where } B_{i_0,\dots,i_j} = (A_{i_0,\dots,i_j} \otimes F^{m*} V_{a_m}) \cap W \subseteq A_{i_0,\dots,i_j} \otimes F^{m*} V_{a_m}$$

and

$$\tilde{C}_j = \bigoplus_{i_0,\dots,i_j} C_{i_0,\dots,i_j}, \text{ where } C_{i_0,\dots,i_j} = (A_{i_0,\dots,i_j} \otimes z^{a_m p^m}) \cap W \subseteq A_{i_0,\dots,i_j} \otimes z^{a_m p^m}.$$

Then B_{i_0,\dots,i_j} and C_{i_0,\dots,i_j} have canonical P -subquotient module structures. Let $\mathcal{B}_{i_0,\dots,i_j}$ and $\mathcal{C}_{i_0,\dots,i_j}$ be the associated G -subbundles in $\mathcal{A}_{i_0,\dots,i_j} \otimes F^{m*} \mathcal{V}_{a_m}$ and $\mathcal{A}_{i_0,\dots,i_j} \otimes \mathcal{O}(a_m p^m)$, respectively. Moreover it follows that $A_{i_0,\dots,i_j} \neq 0$ implies $B_{i_0,\dots,i_j} \neq 0$. The bundle \mathcal{W}' has a filtration by G -subbundles such that subquotients are isomorphic to $\mathcal{B}_{i_0,\dots,i_j}$ or to $\mathcal{C}_{i_0,\dots,i_j}$.

Therefore

$$-\deg \mathcal{W} \geq -\deg \tilde{\mathcal{B}}_{-1} - \deg \tilde{\mathcal{B}}_0 + \cdots - \deg \tilde{\mathcal{B}}_{m-1} - \deg \tilde{\mathcal{C}}_0 + \cdots - \deg \tilde{\mathcal{C}}_{m-1},$$

where $\tilde{\mathcal{B}}_j$ and $\tilde{\mathcal{C}}_j$ are the G -bundles associated to P -modules \tilde{B}_j and \tilde{C}_j .

Henceforth, for a vector bundle \mathcal{B} , we denote rank \mathcal{B} as $|\mathcal{B}|$. Note that $\mathcal{A}_{i_0, \dots, i_j} \otimes F^{*m} \mathcal{V}_{a_m}$ and $\mathcal{A}_{i_0, \dots, i_j} \otimes \mathcal{O}(a_m p^m)$ are semistable bundles on \mathbf{P}_k^n ; as by Lemma 3.5, for $0 \leq a \leq p-1$, the bundle \mathcal{V}_a is semistable, and hence, by Theorem 2.1 of [MR], all Frobenius pullbacks and tensor products of such bundles are semistable.

Now, for $j \geq 0$,

$$\begin{aligned} -\deg \tilde{\mathcal{B}}_j &= - \sum_{i_0, \dots, i_j} \deg \mathcal{B}_{i_0, \dots, i_j} \geq - \sum_{i_0, \dots, i_j} \mu(\mathcal{A}_{i_0, \dots, i_j}) |\mathcal{B}_{i_0, \dots, i_j}| + \sum_{i_0, \dots, i_j} \frac{a_m p^m}{|\mathcal{V}_{a_m}|} |\mathcal{B}_{i_0, \dots, i_j}| \\ &= \sum_{i_0, \dots, i_j} [-(a_0 + \cdots + a_{m-1} p^{m-1}) + a_{i_0} p^{i_0} + \cdots + a_{i_j} p^{i_j}] |\mathcal{B}_{i_0, \dots, i_j}| \\ &\quad + \sum_{i_0, \dots, i_j} \left[\frac{a_{i_0} p^{i_0}}{|\mathcal{V}_{a_{i_0}}|} + \cdots + \frac{a_{i_j} p^{i_j}}{|\mathcal{V}_{a_{i_j}}|} \right] |\mathcal{B}_{i_0, \dots, i_j}| + \sum_{i_0, \dots, i_j} \frac{a_m p^m}{|\mathcal{V}_{a_m}|} |\mathcal{B}_{i_0, \dots, i_j}|, \end{aligned}$$

and, for $j \geq 1$,

$$\begin{aligned} -\deg \tilde{\mathcal{C}}_j &= - \sum_{i_0, \dots, i_j} \deg \mathcal{C}_{i_0, \dots, i_j} \geq - \sum_{i_0, \dots, i_j} \mu(\mathcal{A}_{i_0, \dots, i_j}) |\mathcal{C}_{i_0, \dots, i_j}| - \sum_{i_0, \dots, i_j} a_m p^m |\mathcal{C}_{i_0, \dots, i_j}| \\ &= \sum_{i_0, \dots, i_j} [-(a_0 + \cdots + a_{m-1} p^{m-1}) + a_{i_0} p^{i_0} + \cdots + a_{i_j} p^{i_j}] |\mathcal{C}_{i_0, \dots, i_j}| \\ &\quad + \sum_{i_0, \dots, i_j} \left[\frac{a_{i_0} p^{i_0}}{|\mathcal{V}_{a_{i_0}}|} + \cdots + \frac{a_{i_j} p^{i_j}}{|\mathcal{V}_{a_{i_j}}|} \right] |\mathcal{C}_{i_0, \dots, i_j}| - \sum_{i_0, \dots, i_j} a_m p^m |\mathcal{C}_{i_0, \dots, i_j}|. \end{aligned}$$

By construction, $\tilde{\mathcal{B}}_{-1} = F^{m*} \hat{\mathcal{B}}_{-1} \otimes \mathcal{O}(a_0 + \cdots + a_{m-1} p^{m-1})$, where $\hat{\mathcal{B}}_{-1} \subseteq \mathcal{V}_{a_m}$ is a G -subbundle. By Lemma 3.5, we have $-\deg \hat{\mathcal{B}}_{-1} \geq a_m$. Therefore

$$-\deg \mathcal{B}_{-1} \geq a_m p^m - (a_0 + \cdots + a_{m-1} p^{m-1}) |\tilde{\mathcal{B}}_{-1}|.$$

Similarly

$$\tilde{\mathcal{C}}_0 = \bigoplus_{i_0=0}^{m-1} \mathcal{C}_{i_0} = \bigoplus_{i_0=0}^{m-1} F^{i_0*} \hat{\mathcal{C}}_{i_0} \otimes \mathcal{O}(a_0 + \cdots + \hat{a}_{i_0} p^{i_0} + \cdots + a_m p^m),$$

where $\hat{\mathcal{C}}_{i_0}$ is a G -subbundle of \mathcal{V}_{a_m} . Therefore

$$-\deg \tilde{\mathcal{C}}_0 \geq \sum_{i_0=0}^{m-1} [-(a_0 + \cdots + a_{m-1} p^{m-1}) + a_{i_0} p^{i_0}] |\mathcal{C}_{i_0}| + \sum_{i_0=0}^{m-1} (a_{i_0} p^{i_0} \delta_{i_0} - a_m p^m |\mathcal{C}_{i_0}|),$$

where $\delta_{i_0} = 1$, if $\mathcal{C}_{i_0} \neq 0$, otherwise $\delta_{i_0} = 0$.

Claim 3.9. (1) $\frac{a_m p^m}{|\mathcal{V}_{a_m}|} |\mathcal{B}_{i_0, \dots, i_j}| - a_m p^m |\mathcal{C}_{i_0, \dots, i_j}| \geq 0$.

$$(2) \frac{a_m p^m}{|\mathcal{V}_{a_m}|} |\mathcal{B}_{i_0}| + a_{i_0} p^{i_0} \delta_{i_0} - a_m p^m |\mathcal{C}_{i_0}| \geq a_{i_0} p^{i_0}.$$

$$(3) \frac{a_{i_k} p^{i_k}}{|\mathcal{V}_{a_{i_k}}|} |\mathcal{B}_{i_0, \dots, i_k, \dots, i_j}| \geq a_{i_k} p^{i_k} |\mathcal{B}_{i_0, \dots, \widehat{i_k}, \dots, i_j}|$$

$$(4) \frac{a_{i_k} p^{i_k}}{|\mathcal{V}_{a_{i_k}}|} |\mathcal{C}_{i_0, \dots, i_k, \dots, i_j}| \geq a_{i_k} p^{i_k} |\mathcal{C}_{i_0, \dots, \widehat{i_k}, \dots, i_j}|,$$

where $\mathcal{B}_{i_0, \dots, \widehat{i_k}, \dots, i_j} = \mathcal{B}_{i_0, \dots, i_{k-1}, i_{k+1}, \dots, i_j}$ and $\mathcal{C}_{i_0, \dots, \widehat{i_k}, \dots, i_j} = \mathcal{C}_{i_0, \dots, i_{k-1}, i_{k+1}, \dots, i_j}$

Proof. We note that $C_{i_0, \dots, i_j} = \widehat{C}_{i_0, \dots, i_j} \otimes z^{a_0 + \dots + \widehat{a_{i_0}} p^{i_0} + \dots + \widehat{a_{i_j}} p^{i_j} + \dots + a_m p^m} \subseteq V' \cap W$, where $\widehat{C}_{i_0, \dots, i_j} \subseteq F^{i_0} V_{a_{i_0}} \otimes \dots \otimes F^{i_j} V_{a_{i_j}}$ is a P -submodule. By Remark 3.2, this implies that

$$\widehat{C}_{i_0, \dots, i_j} \otimes F^{m*} V_{a_m} \otimes z^{a_0 + \dots + \widehat{a_{i_0}} p^{i_0} + \dots + \widehat{a_{i_j}} p^{i_j} + \dots + a_{m-1} p^{m-1}} \subseteq V' \cap W,$$

therefore

$$\widehat{C}_{i_0, \dots, i_j} \otimes F^{m*} V_{a_m} \otimes z^{a_0 + \dots + \widehat{a_{i_0}} + \dots + \widehat{a_{i_j}} + \dots + a_{m-1} p^{m-1}} \subseteq B_{i_0, \dots, i_j}.$$

Hence

$$(3.1) \quad |B_{i_0, \dots, i_j}| \geq |\widehat{C}_{i_0, \dots, i_j}| |V_{a_m}| = |C_{i_0, \dots, i_j}| |V_{a_m}|.$$

This proves assertion (1).

Similarly one can check that

$$|B_{i_0, \dots, i_k, \dots, i_j}| \geq |V_{a_{i_k}}| |B_{i_0, \dots, \widehat{i_k}, \dots, i_j}| \text{ and } |C_{i_0, \dots, \widehat{i_k}, \dots, i_j}| \geq |V_{a_{i_k}}| |C_{i_0, \dots, i_k, \dots, i_j}|.$$

Hence assertions (3) and (4) follow.

Now, assertion (2) follows, by inequality (3.1), if $|C_{i_0}| \neq 0$. Therefore we can assume that $|C_{i_0}| = 0$. We can also assume that $a_{i_0} \neq 0$. By Remark 3.2,

$$W(a_0 + \dots + a_{m-1} p^{m-1}) \neq 0 \implies W(a_0 + \dots + a_{i_0-1} p^{i_0-1} + a_{i_0+1} p^{i_0+1} + \dots + a_{m-1} p^{m-1}) \neq 0.$$

Therefore, by Lemma 3.3,

$$F^{i_0} S^{a_{i_0}}(V_1) \otimes F^{m*} S^{a_m}(V_1) \otimes z^{a_0 + \dots + \widehat{a_{i_0}} p^{i_0} + \dots + a_{m-1} p^{m-1}} \subseteq B_{i_0}.$$

Hence $|B_{i_0}| \geq |S^{a_{i_0}}(V_1)| |S^{a_m}(V_1)|$. But, for any interger $a \geq 1$, we can check the inequality

$$\frac{|S^a(V_1)|}{|\mathcal{V}_a|} \geq \frac{n}{n+a}.$$

Therefore

$$\begin{aligned} \frac{a_m p^m}{|\mathcal{V}_{a_m}|} |B_{i_0}| &\geq \frac{a_m p^m}{|\mathcal{V}_{a_m}|} |S^{a_{i_0}}(V_1)| |S^{a_m}(V_1)| \\ &\geq \frac{n a_m p^m}{(n+a_m)} |S^{a_{i_0}}(V_1)| \geq \frac{n a_m p}{(n+a_m)} p^{m-1} |S^{a_{i_0}}(V_1)| \geq a_{i_0} p^{i_0}, \end{aligned}$$

where the last inequality follows as $i_0 \leq m-1$. This proves the assertion (2), and hence the claim.

Therefore

$$-\deg \widetilde{\mathcal{B}}_{-1} - \deg \widetilde{\mathcal{B}}_0 - \deg \widetilde{\mathcal{C}}_0 = [a_m p^m] + \left[-(a_0 + \dots + a_{m-1} p^{m-1}) |\widetilde{\mathcal{B}}_{-1}| + \sum_{i_0=0}^{m-1} \frac{a_{i_0} p^{i_0}}{|\mathcal{V}_{a_{i_0}}|} |\mathcal{B}_{i_0}| \right]$$

$$+ \sum_{i_0=0}^{m-1} [-(a_0 + \cdots + a_{m-1}p^{m-1}) + a_{i_0}p^{i_0}] [|\mathcal{B}_{i_0}| + |\mathcal{C}_{i_0}|] + \sum_{i_0=0}^{m-1} \left[\frac{a_m p^m}{|V_{a_m}|} |\mathcal{B}_{i_0}| - a_m p^m |\mathcal{C}_{i_0}| + a_{i_0} p^{i_0} \delta_{i_0} \right].$$

Now, by assertions (3) and (2) of Claim 3.9, applied respectively to the terms in second and fourth bracket above, we get

$$\begin{aligned} & -\deg \tilde{\mathcal{B}}_{-1} - \deg \tilde{\mathcal{B}}_0 - \deg \tilde{\mathcal{C}}_0 \\ & \geq a_0 + \cdots + a_m p^m + \sum_{i_0=0}^{m-1} [-(a_0 + \cdots + a_{m-1}p^{m-1}) + a_{i_0}p^{i_0}] [|\mathcal{B}_{i_0}| + |\mathcal{C}_{i_0}|]. \end{aligned}$$

Moreover, applying assertion (1) of Claim 3.9, we get

$$\begin{aligned} -\deg \tilde{\mathcal{B}}_1 - \deg \tilde{\mathcal{C}}_1 &= \sum_{i_0, i_1} [-(a_0 + \cdots + a_{m-1}p^{m-1}) + a_{i_0}p^{i_0} + a_{i_1}p^{i_1}] + \\ & \sum_{i_0, i_1} \left[\frac{a_{i_0}p^{i_0}}{|V_{a_{i_0}}|} + \frac{a_{i_1}p^{i_1}}{|V_{a_{i_1}}|} \right] [|\mathcal{B}_{i_0 i_1}| + |\mathcal{C}_{i_0 i_1}|]. \end{aligned}$$

Claim 3.10.

$$\sum_{i_0, i_1} \left[\frac{a_{i_0}p^{i_0}}{|V_{a_{i_0}}|} + \frac{a_{i_1}p^{i_1}}{|V_{a_{i_1}}|} \right] [|\mathcal{B}_{i_0 i_1}| + |\mathcal{C}_{i_0 i_1}|] \geq \sum_{i_0=0}^{m-1} [(a_0 + \cdots + a_{m-1}p^{m-1}) - a_{i_0}p^{i_0}] [|\mathcal{B}_{i_0}| + |\mathcal{C}_{i_0}|].$$

Proof. By assertion (3) and (4) of Claim 3.9, we have

$$\begin{aligned} \text{left hand side} &\geq \sum_{i_0, i_1} [a_{i_0}p^{i_0}(|\mathcal{B}_{i_1}| + |\mathcal{C}_{i_1}|)] + \sum_{i_0, i_1} [a_{i_1}p^{i_1}(|\mathcal{B}_{i_0}| + |\mathcal{C}_{i_0}|)] \\ &= \sum_{i_0, i_1} [a_{i_0}p^{i_0}|\mathcal{B}_{i_1}| + a_{i_1}p^{i_1}|\mathcal{B}_{i_0}|] + \sum_{i_0, i_1} [a_{i_0}p^{i_0}|\mathcal{C}_{i_1}| + a_{i_1}p^{i_1}|\mathcal{C}_{i_0}|] \\ &= \sum_{i_0=0}^{m-1} [(a_0 + \cdots + a_{m-1}p^{m-1}) - a_{i_0}p^{i_0}] [|\mathcal{B}_{i_0}| + |\mathcal{C}_{i_0}|]. \end{aligned}$$

This proves the claim.

In particular

$$\begin{aligned} & -\deg \tilde{\mathcal{B}}_{-1} - \deg \tilde{\mathcal{B}}_0 - \deg \tilde{\mathcal{C}}_0 - \deg \tilde{\mathcal{B}}_1 - \deg \tilde{\mathcal{C}}_1 \geq \\ & (a_0 + \cdots + a_m p^m) + \sum_{i_0, i_1} [-(a_0 + \cdots + a_{m-1}p^{m-1}) + a_{i_0}p^{i_0} + a_{i_1}p^{i_1}] [|\mathcal{B}_{i_0 i_1}| + |\mathcal{C}_{i_0 i_1}|]. \end{aligned}$$

By assertion (1) of Claim 3.9, for $j \geq 2$, we have

$$\begin{aligned} & -\deg \tilde{\mathcal{B}}_j - \deg \tilde{\mathcal{C}}_j = \sum_{i_0, \dots, i_j} \left[\frac{a_{i_0}p^{i_0}}{|V_{a_{i_0}}|} + \cdots + \frac{a_{i_j}p^{i_j}}{|V_{a_{i_j}}|} \right] [|\mathcal{B}_{i_0, \dots, i_j}| + |\mathcal{C}_{i_0, \dots, i_j}|] \\ & + \sum_{i_0, \dots, i_j} [-(a_0 + \cdots + a_{m-1}p^{m-1}) + a_{i_0}p^{i_0} + \cdots + a_{i_j}p^{i_j}] [|\mathcal{B}_{i_0, \dots, i_j}| + |\mathcal{C}_{i_0, \dots, i_j}|] \end{aligned}$$

Now one can check (similar to the proof of Claim 3.10) that

$$\sum_{i_0, \dots, i_j} \left[\frac{a_{i_0} p^{i_0}}{|V_{a_{i_0}}|} + \dots + \frac{a_{i_j} p^{i_j}}{|V_{a_{i_j}}|} \right] [|\mathcal{B}_{i_0, \dots, i_j}| + |\mathcal{C}_{i_0, \dots, i_j}|] \geq$$

$$\sum_{i_0, \dots, i_{j-1}} [(a_0 + \dots + a_{m-1} p^{m-1}) - (a_{i_0} p^{i_0} + \dots + a_{i_{j-1}} p^{i_{j-1}})] [|\mathcal{B}_{i_0, \dots, i_{j-1}}| + |\mathcal{C}_{i_0, \dots, i_{j-1}}|].$$

Now it follows that, for $0 \leq j \leq m-1$,

$$-\deg \tilde{\mathcal{B}}_{-1} - (\deg \tilde{\mathcal{B}}_0 + \deg \tilde{\mathcal{C}}_0) - \dots - (\deg \tilde{\mathcal{B}}_j + \deg \tilde{\mathcal{C}}_j) \geq (a_0 + \dots + a_m p^m)$$

$$+ \sum_{i_0, \dots, i_j} [-(a_0 + \dots + a_{m-1} p^{m-1}) + a_{i_0} p^{i_0} + \dots + a_{i_j} p^{i_j}] [|\mathcal{B}_{i_0 \dots i_j}| + |\mathcal{C}_{i_0 \dots i_j}|].$$

Therefore, for $j = m-1$

$$-\deg \tilde{\mathcal{B}}_{-1} - (\deg \tilde{\mathcal{B}}_0 + \deg \tilde{\mathcal{C}}_0) - \dots - (\deg \tilde{\mathcal{B}}_{m-1} + \deg \tilde{\mathcal{C}}_{m-1}) \geq a_0 + \dots + a_m p^m.$$

This proves the lemma. \square

We have immediate corollary of this lemma.

Corollary 3.11. *Suppose $d = a_0 + a_m p^m$ is the p -adic expansion of the positive integer d . Suppose $\mathcal{W} \subset \mathcal{V}_d$ is a G -subbundle such that $W(a_0) \neq 0$. Then $-\deg(\mathcal{W}) \geq d = a_0 + a_m p^m$.*

Lemma 3.12. *Suppose $\mathcal{W} \subset \mathcal{V}_d$ is a G -subbundle such that $W(a_0) \neq 0$. Let i be the integer such that $W(a_0 + \dots + a_{i-1} p^{i-1}) \neq 0$ and $W(a_0 + \dots + a_i p^i) = 0$. We further assume that a_{i+1}, \dots, a_m are nonzero integers. Then, there exists $i \leq k \leq m$, such that*

$$-\deg \mathcal{W} \geq (a_0 + \dots + a_k p^k) [(h(a_{k+1})) (h(a_{k+2}) - 1) \dots (h(a_m) - 1)],$$

where $h(t) = \dim H^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(t))$.

Proof. By hypothesis it follows that $1 \leq i \leq m$. If $i = m$ then, by Lemma 3.8, $-\deg \mathcal{W} \geq a_0 + a_1 p + \dots + a_m p^m$. Hence we can assume that $1 \leq i \leq m-1$.

Let $f : H^0(\mathcal{O}(a_0 + \dots + a_{m-1} p^{m-1})) \otimes F^{*m}(\mathcal{V}_{a_m}) \rightarrow \mathcal{V}_{a_0 + \dots + a_m p^m}$ be the canonical map. As in the proof of Lemma 3.8, we get a commutative diagram of G -bundles

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ 0 \rightarrow & \mathcal{V}_{(\oplus_{i=0}^{m-1} a_i p^i)} \otimes F^{*m} \mathcal{V}_{a_m} & \rightarrow & H^0(\mathcal{O}(\oplus_{i=0}^{m-1} a_i p^i)) \otimes F^{*m} \mathcal{V}_{a_m} & \rightarrow & \mathcal{O}(\oplus_{i=0}^{m-1} a_i p^i) \otimes F^{*m} \mathcal{V}_{a_m} & \rightarrow 0 \\ & & & \downarrow & & & \\ & & & \mathcal{V}_{a_0 + \dots + a_m p^m} & & & \\ & & & \downarrow & & & \\ 0 \rightarrow & \mathcal{V}_{(\oplus_{i=0}^{m-1} a_i p^i)} \otimes \mathcal{O}(a_m p^m) & \rightarrow & \text{coker}(f) & \rightarrow & \oplus \mathcal{O}_X & \rightarrow 0 \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

We note that, if \mathcal{V}' denotes the homogeneous subbundle given as $\mathcal{V}' = \text{kernel}$ of the canonical composite map

$$\mathcal{V}_{a_0+\dots+a_m p^m} \longrightarrow \text{coker}(f) \longrightarrow \oplus \mathcal{O}_X$$

and $\mathcal{W}' = \mathcal{W} \cap \mathcal{V}'$, then $-\deg \mathcal{W} \geq -\deg \mathcal{W}'$. Since $i \leq m-1$, we have $W(a_0 + \dots + a_{m-1}p^{m-1}) = 0$. Therefore

$$\mathcal{W}' \subseteq \mathcal{V}_{a_0+\dots+a_{m-1}p^{m-1}} \otimes H^0(\mathcal{O}(a_m))^{(p^m)},$$

as G -subbundle. Hence, using Lemma 3.1, we see that \mathcal{W}' has a G -stable filtration with

$$\text{gr } \mathcal{W}' = \bigoplus_{i=0}^{a_m} \mathcal{A}_{0i} \otimes F^{*m}(S^{a_m-i}(\mathcal{V}_1) \otimes \mathcal{O}(i)),$$

where $\mathcal{A}_{0i} \subseteq \mathcal{V}_{a_0+\dots+a_{m-1}p^{m-1}}$ is a homogeneous G -subbundle. Note that, by Remark 3.2 and Lemma 3.3, we have $\mathcal{A}_{00} \neq 0$. We have $-\deg \mathcal{W} \geq -\deg \mathcal{W}'$, where

$$\begin{aligned} -\deg \mathcal{W}' &= \sum_{i=0}^{a_m} -(\deg \mathcal{A}_{0i})|S^{a_m-i}(\mathcal{V}_1)| - \sum_{i=0}^{a_m} |\mathcal{A}_{0i}| \deg F^{*m}(S^{a_m-i}(\mathcal{V}_1) \otimes \mathcal{O}(i)) \\ &= \sum_{i=0}^{a_m} -(\deg \mathcal{A}_{0i})|S^{a_m-i}(\mathcal{V}_1)| - p^m \sum_{i=0}^{a_m} |\mathcal{A}_{0i}| |S^{a_m-i}(\mathcal{V}_1)| [\mu(S^{a_m-i}(\mathcal{V}_1)) + \mu(\mathcal{O}(i))] \\ (3.2) \quad & -\deg \mathcal{W}' = \sum_{i=0}^{a_m} -(\deg \mathcal{A}_{0i})|S^{a_m-i}(\mathcal{V}_1)| + \frac{p^m}{n} \sum_{i=0}^{a_m} |\mathcal{A}_{0i}| |S^{a_m-i}(\mathcal{V}_1)| (a_m - i - ni). \end{aligned}$$

Note that, by induction on m we have $-\deg \mathcal{A}_{0i} \geq 0$, for $0 \leq i \leq a_m$. Let

$$(\star) = \sum_{i=0}^{a_m} |\mathcal{A}_{0i}| |S^{a_m-i}(\mathcal{V}_1)| (a_m - i - ni)$$

Now applying the identity $(a+1)|S^{a+1}(\mathcal{V}_1)| = n(|S^a(\mathcal{V}_1)| + \dots + |S^0(\mathcal{V}_1)|)$, we have

$$\begin{aligned} (\star) &= n|S^{a_m-1}(\mathcal{V}_1)|(|\mathcal{A}_{00}| - |\mathcal{A}_{01}|) \\ &\quad + n|S^{a_m-2}(\mathcal{V}_1)|(|\mathcal{A}_{00}| - |\mathcal{A}_{02}| + |\mathcal{A}_{01}| - |\mathcal{A}_{02}|) \\ &\quad \vdots \\ &\quad + n|S^1(\mathcal{V}_1)|(|\mathcal{A}_{00}| - |\mathcal{A}_{0(a_m-1)}| + \dots + |\mathcal{A}_{0(a_m-2)}| - |\mathcal{A}_{0(a_m-1)}|) + \\ &\quad n|S^0(\mathcal{V}_1)|(|\mathcal{A}_{00}| - |\mathcal{A}_{0(a_m)}| + \dots + |\mathcal{A}_{0(a_m-2)}| - |\mathcal{A}_{0(a_m)}| + |\mathcal{A}_{0(a_m-1)}| - |\mathcal{A}_{0(a_m)}|) \end{aligned}$$

Note that each term $|\mathcal{A}_{0i}| - |\mathcal{A}_{0j}|$ (with $j > i$) is always non-negative, as by Lemma 3.1 (2), we have $\mathcal{A}_{0j} \subseteq \mathcal{A}_{0i}$.

Case (1) If $W(a_m p^m) = 0$, then $\mathcal{A}_{0a_m} = 0$; now we choose $1 \leq j \leq a_m$ such that $\mathcal{A}_{0j} = 0$ and $\mathcal{A}_{0(j-1)} \neq 0$. Then

$$\begin{aligned} (\star) &\geq n|S^{a_m-j}(\mathcal{V}_1)|(|\mathcal{A}_{00}| - |\mathcal{A}_{0j}| + \dots + |\mathcal{A}_{0(j-1)}| - |\mathcal{A}_{0j}|) + \\ &\quad \dots + n|S^0(\mathcal{V}_1)|(|\mathcal{A}_{00}| - |\mathcal{A}_{0a_m}| + \dots + |\mathcal{A}_{0(a_m-1)}| - |\mathcal{A}_{0a_m}|) \\ &= n(|\mathcal{A}_{00}| + \dots + |\mathcal{A}_{0(j-1)}|)(|S^{a_m-j}(\mathcal{V}_1)| + \dots + |S^0(\mathcal{V}_1)|) \geq n^2 \cdot j \cdot h^0(\mathcal{O}(a_m - j)). \end{aligned}$$

Therefore

$$\frac{p^m}{n}(\star) \geq p^m n j h^0(\mathcal{O}(a_m - j)) \geq p^m(a_m + 1) \geq a_0 + \cdots + a_m p^m.$$

Case (2). Suppose $W(a_m p^m) \neq 0$, so that $\mathcal{A}_{0i} \neq 0$, for all i . Choose i to be the largest integer such that $|A_{00}| = \cdots = |A_{0i}|$.

(a) If $i = a_m$ then $(\star) = 0$, and so Equation 3.2 becomes

$$-\deg \mathcal{W} \geq -(\deg \mathcal{A}_{00})h^0(\mathcal{O}(a_m)).$$

(b) If $i = a_m - 1$, then

$$-\deg \mathcal{W} \geq -(\deg \mathcal{A}_{00})(h^0(\mathcal{O}(a_m)) - 1) + a_m p^m.$$

(c) If $i < a_m - 1$ then

$$\frac{p^m}{n}(\star) \geq p^m h^0(\mathcal{O}(a_m - (i + 1)))(i + 1) \geq (a_m + 1)p^m.$$

Hence we conclude, from case (1) and case (2) that either

$$-\deg \mathcal{W} \geq a_0 + \cdots + a_m p^m \text{ or } -\deg \mathcal{W} \geq -(\deg \mathcal{A}_{00})(h(a_m) - 1) + \delta_m,$$

where $\delta_m = \min\{-\deg \mathcal{A}_{00}, a_m p^m\}$. Let

$$\mathcal{A}_{m-1} := \mathcal{A}_{00} \subseteq \mathcal{V}_{a_0 + \cdots + a_{m-1} p^{m-1}}.$$

Note that

$$\mathcal{W}(a_0) \neq 0 \implies \mathcal{W}'(a_0) \neq 0 \implies \mathcal{A}_{00} \neq 0, \text{ i.e. } \mathcal{A}_{m-1}(a_0) \neq 0.$$

Then by replacing \mathcal{W} , \mathcal{W}' and \mathcal{V}_d by the G -homogeneous bundles \mathcal{A}_{m-1} , \mathcal{A}'_{m-1} and $\mathcal{V}_{a_0 + \cdots + a_{m-1} p^{m-1}}$ respectively we have

$$\text{gr } \mathcal{A}'_{m-1} = \bigoplus_{i=0}^{a_{m-1}} \mathcal{A}_{1i} \otimes F^{*m-1}(S^{a_{m-1}-i}(\mathcal{V}_1) \otimes \mathcal{O}(i)),$$

where $\mathcal{A}_{1i} \subseteq \mathcal{V}_{a_0 + \cdots + a_{m-2} p^{m-2}}$, for each i , is a G -homogeneous subbundle. Then either

$$-\deg \mathcal{A}_{m-1} \geq a_0 + a_1 p + \cdots + a_{m-1} p^{m-1}$$

or

$$-(\deg \mathcal{A}_{m-1}) \geq -(\deg \mathcal{A}_{m-2})(h(a_{m-1}) - 1) + \delta_{m-1},$$

where $\delta_{m-1} = \min\{-\deg \mathcal{A}_{m-2}, a_{m-1} p^{m-1}\}$.

Now inductively define $\mathcal{A}_{m-i} = \mathcal{A}_{(i-1)0}$. Equivalently, we define \mathcal{A}_{j-1} as a subset of $V_{a_0 + a_1 p + \cdots + a_{j-1} p^{j-1}}$ such that

$$\begin{aligned} & \mathcal{A}_{j-1} \otimes F^{*j} S^{a_j}(V_1) \otimes \cdots \otimes F^{*m} S^{a_m}(V_1) \\ &= (V_{a_0 + \cdots + a_{j-1} p^{j-1}} \otimes F^{*j} S^{a_j}(V_1) \otimes \cdots \otimes F^{*m} S^{a_m}(V_1)) \cap W. \end{aligned}$$

Let $\delta_j = \min\{-\deg \mathcal{A}_{j-1}, a_j p^j\}$.

Let k be the largest integer such that $-\deg \mathcal{A}_k \geq a_0 + \cdots + a_k p^k$. By Lemma 3.8, we have $k \geq i$. Now, for every $l \geq k$, we have

$$-\deg \mathcal{A}_l \geq -\deg \mathcal{A}_{l+1}(h(a_l) - 1) + \delta_l.$$

This implies that

$$-\deg \mathcal{W} \geq -\deg \mathcal{A}_k [(h(a_{k+1}) - 1)(h(a_{k+2}) - 1) \cdots (h(a_m) - 1)] \\ + \delta_{k+1} [(h(a_{k+2}) - 1) \cdots (h(a_m) - 1)] + \delta_{k+2} [(h(a_{k+3}) - 1) \cdots (h(a_m) - 1)] + \cdots + \delta_m.$$

Since

$$\delta_{k+1} \geq \min\{a_0 + \cdots + a_k p^k, a_{k+1} p^{k+1}\} \geq a_0 + a_1 p + \cdots + a_k p^k,$$

as $a_{k+1} \neq 0$, we have

$$-\deg \mathcal{W} \geq (a_0 + \cdots + a_k p^k) [(h(a_{k+1})) (h(a_{k+2}) - 1) \cdots (h(a_m) - 1)].$$

This proves the lemma. \square

Remark 3.13. We can generalise the statement of Lemma 3.12 as follows: Let $\mathcal{W} \subset \mathcal{V}_d$ be a G -subbundle such that $W(a_0) \neq 0$. Let i be the integer such that $W(a_0 + \cdots + a_{i-1} p^{i-1}) \neq 0$ and $W(a_0 + \cdots + a_i p^i) = 0$. Then, there exists $i \leq k \leq m$, such that

$$-\deg \mathcal{W} \geq (a_0 + \cdots + a_k p^k) [|S^{a_{k+1}}(V_1)| |S^{a_{k+2}}(V_1)| \cdots |S^{a_m}(V_1)|].$$

This follows from the argument given, in the above proof, that always (independent of the fact that some $a_j = 0$ or $\neq 0$), we have

$$-\deg \mathcal{W} \geq a_0 + \cdots + a_m p^m \text{ or } -\deg \mathcal{W} \geq (-\deg \mathcal{A}_{00}) |S^{a_m}(V_1)|.$$

Now, by induction on m , the result follows.

Corollary 3.14. *If $n \geq d/p$, then, for any G -subbundle $\mathcal{W} \subset \mathcal{V}_d$, we have $-\deg \mathcal{W} > d$.*

Proof. By Lemma 3.7, we can assume that $W(a_0) \neq 0$. By Corollary 3.11, we can also assume that $m \geq 2$. Now, if $-\deg \mathcal{W} \not\geq a_0 + \cdots + a_m p^m$, then, by Remark 3.13, there exists an integer k with $1 \leq k \leq m-1$ such that

$$-\deg \mathcal{W} \geq (a_0 + \cdots + a_k p^k) (|S^{a_{k+1}}(V_1)| \cdots |S^{a_m}(V_1)|),$$

where $a_k \neq 0$ and $k \geq 1$, which implies

$$-\deg \mathcal{W} \geq (a_0 + a_k p^k) n \geq (a_0 + a_k p^k) (d/p) > d.$$

This proves the corollary. \square

Remark 3.15. If along with the hypothesis of Lemma 3.12, we have the additional conditions, namely $a_{k+1} = \cdots, a_{m-1} = 1$ and $p \leq n$, then it is easy to see that

$$-\deg \mathcal{W} \geq (a_0 + \cdots + a_k p^k) h(a_{k+1}) h(a_{k+2}) \cdots h(a_m).$$

Lemma 3.16. *Let $\mathcal{W} \subset \mathcal{V}_d$ be a G -subbundle such that $W(a_0) \neq 0$. If $p \leq n$ and $a_2, \dots, a_m \geq 1$ then, $-\deg \mathcal{W} \geq d$.*

Proof. By Lemma 3.7, we can assume that $W(a_0) \neq 0$. Therefore, by Lemma 3.12, if $-\deg \mathcal{W} \not\geq a_0 + \cdots + a_m p^m$, then there exists $1 \leq k \leq m-1$ such that

$$-\deg \mathcal{W} \geq (a_0 + \cdots + a_k p^k) h(a_{k+1}) (h(a_{k+2}) - 1) \cdots (h(a_m) - 1)$$

Moreover if $a_{k+1} = \dots = a_m = 1$ then, by Remark 3.15, we have

$$-\deg \mathcal{W} \geq (a_0 + \dots + a_k p^k) h(a_{k+1}) h(a_{k+2}) \dots h(a_m).$$

In this case

$$\begin{aligned} -\deg \mathcal{W} &\geq (a_0 + \dots + a_k p^k) h(a_{k+1}) \dots h(a_m) \\ &\geq (a_0 + \dots + a_k p^k) (n+1)^{m-k} \\ &\geq (a_0 + \dots + a_k p^k) (p+1)^{m-k}, \end{aligned}$$

which implies that $-\deg \mathcal{W} \geq a_0 + \dots + a_m p^m$.

If $a_t \geq 2$ for some $k < t \leq m$. Then $h(a_t) - 1 \geq n(n+3)$. Therefore

$$\begin{aligned} -\deg \mathcal{W} &\geq (a_0 + \dots + a_k p^k) a_{k+1} \dots \hat{a}_t \dots a_m (n(n+3)) n^{m-k-1} \\ &\geq a_k p^m (a_m + 4) \geq a_0 + a_1 p + \dots + a_m p^m. \end{aligned}$$

This proves the lemma. \square

Lemma 3.17. *Let $\mathcal{W} \subseteq \mathcal{V}_d$ be a G -subbundle such that $p \geq n$ and $W(a_0) \neq 0$. Let $1 \leq i \leq m-1$ be a nonnegative integer such that $W(a_0 + \dots + a_{i-1} p^{i-1}) \neq 0$ and $W(a_0 + \dots + a_i p^i) = 0$. Moreover*

$$a_{i+1}, \dots, a_{m-1}, a_m \in \{p-n+1, \dots, p-1\}.$$

Then $-\deg \mathcal{W} \geq d$.

Proof. By Lemma 3.7, we can assume that $W(a_0) \neq 0$. Moreover, by Lemma 3.16, we can assume that $p > n$. If $\deg \mathcal{W} \geq a_0 + \dots + a_m p^m$ then the lemma follows. Hence, by Lemma 3.12, there exists an integer k such that $i \leq k \leq m-1$ and

$$-\deg \mathcal{W} \geq (a_0 + \dots + a_k p^k) h(a_{k+1}) (h(a_{k+2}) - 1) \dots (h(a_m) - 1).$$

Note that, by hypothesis, $a_m \geq p-n+1$, which implies that $h(a_m) \geq (p+1)a_m$. Therefore

$$-\deg \mathcal{W} \geq (a_k p^k) h(a_{k+1}) (h(a_{k+2}) - 1) \dots (h(a_m) - 1).$$

(1) If $1 \leq k \leq m-2$ then

$$\begin{aligned} -\deg \mathcal{W} &\geq (a_k p^k) h(a_{k+1}) (h(a_{k+2}) - 1) \dots (h(a_m) - 1) \\ &\geq a_k p^k \binom{p-n+1+n}{n} \binom{p-n+1+n-1}{n-1}^{m-2-k} (pa_m) \\ &\geq a_k p^k \frac{p(p+1)}{2} \left(\frac{p(p-1)}{2} \right)^{m-2-k} (pa_m) \\ &\geq a_m p^m \frac{(p+1)}{2} \text{ as } p > n \geq 3 \\ &\geq a_0 + \dots + a_m p^m. \end{aligned}$$

(2) If $k = m-1$ then

$$\begin{aligned} -\deg \mathcal{W} &\geq (a_0 + \dots + a_{m-1} p^{m-1}) h(a_m) \\ &\geq (a_0 + \dots + a_{m-1} p^{m-1}) ((p+1)a_m) \\ &\geq a_0 + \dots + a_m p^m = d. \end{aligned}$$

This proves the lemma. \square

Remark 3.18. Let \mathcal{V}_d be a bundle on \mathbf{P}_k^n with $n \geq 3$ and let $\mathcal{W} \subseteq \mathcal{V}$ be a G -subbundle. Then we have proved

- (1) if $W(a_0) = 0$, then, by Lemma 3.7,

$$-\deg \mathcal{W} \geq (|C_0(\mathcal{W})| + \cdots + |C_{m-1}(\mathcal{W})|) + n|C_m(\mathcal{W})|,$$

where $C_i(\mathcal{W})$ is defined as in Remark 3.6,

- (2) if $W(a_0) \neq 0$ and satisfies the hypothesis of Corollary 3.11, Corollary 3.14, Lemma 3.16 or Lemma 3.17, then

$$-\deg \mathcal{W} \geq a_0 + a_1p + \cdots + a_mp^m.$$

In other words if $\mathcal{W} \subseteq \mathcal{V}_d$ is homogeneous G -subbundle satisfying the hypothesis of Corollary 3.11, Corollary 3.14, Lemma 3.16 or Lemma 3.17, then, either

- (1) $-\deg \mathcal{W} \geq a_0 + a_1p + \cdots + a_mp^m$ or
 (2) $-\deg \mathcal{W} \geq (|C_0(\mathcal{W})| + \cdots + |C_{m-1}(\mathcal{W})|) + n|C_m(\mathcal{W})|.$

4. MAIN RESULTS

Proof of Theorem 1.2. If $\mathbf{P}_k^n = \mathbf{P}_k^2$ then stability of the bundle \mathcal{V}_d follows by Proposition 2.1. Hence the statement (1).

Therefore we can assume that $n \geq 3$. Let $\mathcal{W} \subseteq \mathcal{V}_d$ be a homogenous subbundle. Let $d = (a_0 + a_1p + \cdots + a_mp^m)p^{i_0}$ be the integer satisfying the hypotheses of the theorem. We write a_j as b_{i_0+j} , in particular we write $d = (b_{i_0} + \cdots + b_{i_0+m}p^m)p^{i_0}$, where b_{i_0} and b_{i_0+m} are positive integers. Consider the following diagram of G -bundles:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & F^{*i_0}(\mathcal{V}_{d'}) & \rightarrow & H^0(\mathcal{O}(d'))^{(p^{i_0})} \otimes \mathcal{O}_{\mathbf{P}_k^n} & \rightarrow & \mathcal{O}(d'p^{i_0}) \rightarrow 0 \\ & & \downarrow f & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{V}_{d'p^{i_0}} & \rightarrow & H^0(\mathcal{O}(d'p^{i_0})) \otimes \mathcal{O}_{\mathbf{P}_k^n} & \rightarrow & \mathcal{O}(d'p^{i_0}) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{coker}(f) & & \oplus \mathcal{O}_{\mathbf{P}_k^n} & & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where $d' = a_0 + \cdots + a_mp^m = b_{i_0} + \cdots + b_{i_0+m}p^m$ and $H^0(\mathcal{O}(a))$ denotes $H^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(a))$. Therefore, for any G -subbundle $\mathcal{W} \subseteq \mathcal{V}_d$, we have

$$-\deg \mathcal{W} \geq -\deg (\mathcal{W} \cap F^{*i_0}(\mathcal{V}_{b_{i_0} + \cdots + b_{i_0+m}p^m})).$$

Then, by Lemma 3.1, we have $\mathcal{W} \cap F^{*i_0}(\mathcal{V}_{b_{i_0} + \cdots + b_{i_0+m}p^m}) = F^{*i_0}(\mathcal{W}_1)$, for some G -subbundle $\mathcal{W}_1 \subseteq \mathcal{V}_{b_{i_0} + \cdots + b_{i_0+m}p^m}$ and corresponding P -submodule $W_1 \subseteq V_{b_{i_0} + \cdots + b_{i_0+m}p^m}$.

Now, if for some

$$j_0 + j_1p + \cdots + j_{i_0-1}p^{i_0-1} + j_{i_0}p^{i_0} + \cdots + j_{i_0+m}p^{i_0+m} < b_{i_0}p^{i_0} + \cdots + b_{i_0+m}p^{i_0+m}$$

we have

$$W(j_0 + j_1p + \cdots + j_{i_0-1}p^{i_0-1} + j_{i_0}p^{i_0} + \cdots + j_{i_0+m}p^{i_0+m}) \neq 0$$

then, by part (1) of Remark 3.2,

$$W(j_{i_0}p^{i_0} + \cdots + j_{i_0+m}p^{i_0+m}) \neq 0,$$

and hence by part (2) of Remark 3.2,

$$W_1(j_{i_0} + \cdots + j_{i_0+m}p^m) \neq 0.$$

Therefore, if

$$\begin{aligned} C_{i_0}(\mathcal{W}) &= \{(j_0, \dots, j_{i_0}, b_{i_0+1}, \dots, b_{i_0+m}) \mid 0 \leq j_{i_0} < b_{i_0}, \text{ and } W_{j_0, \dots, j_{i_0}, b_{i_0+1}, \dots, b_{i_0+m}} \neq 0\} \\ &\vdots \\ C_{i_0+m}(\mathcal{W}) &= \{(j_0, \dots, j_{i_0-1}, j_{i_0}, \dots, j_{i_0+m}) \mid 0 \leq j_{i_0+m} < b_{i_0+m}, \text{ and } W_{j_0, \dots, j_{i_0+m}} \neq 0\} \end{aligned}$$

Then, $C_k(\mathcal{W}) = \phi$, for $k < i_0$, and therefore

$$\text{gr } \mathcal{W} = \bigoplus_{(j_0, \dots, j_{i_0+m}) \in C_{i_0}(\mathcal{W}) \cup \cdots \cup C_{i_0+m}(\mathcal{W})} \mathcal{W}_{j_0, \dots, j_{i_0+m}}$$

and by Remark 3.6,

$$-\mu(\mathcal{V}_d)|W| \leq (|C_{i_0}(\mathcal{W})| + \cdots + |C_{i_0+m-1}(\mathcal{W})|) + n|C_{i_0+m}(\mathcal{W})|.$$

On the other hand

$$\begin{aligned} C_0(\mathcal{W}_1) &= \{(j_{i_0}, b_{i_0+1}, \dots, b_{i_0+m}) \mid 0 \leq j_{i_0} < b_{i_0}, \text{ and } (W_1)_{j_{i_0}, b_{i_0+1}, \dots, b_{i_0+m}} \neq 0\} \\ &\vdots \\ C_t(\mathcal{W}_1) &= \{(j_{i_0}, \dots, j_{i_0+t}, b_{i_0+(t+1)}, \dots, b_{i_0+m}) \mid 0 \leq j_{i_0+t} < b_{i_0+t}, \\ &\quad \text{and } (W_1)_{j_{i_0}, \dots, j_{i_0+t}, b_{i_0+(t+1)}, \dots, b_{i_0+m}} \neq 0\} \\ &\vdots \\ C_m(\mathcal{W}_1) &= \{(j_{i_0}, \dots, j_{i_0+m}) \mid 0 \leq j_{i_0+m} < b_{i_0+m}, \text{ and } (W_1)_{j_{i_0}, \dots, j_{i_0+m}} \neq 0\} \end{aligned}$$

then

$$|C_{i_0}(\mathcal{W})| \leq p^{i_0}|C_0(\mathcal{W}_1)|, \dots, |C_{i_0+m}(\mathcal{W})| \leq p^{i_0}|C_m(\mathcal{W}_1)|.$$

Since $\mathcal{W}_1 \subseteq \mathcal{V}_{b_{i_0} + \cdots + b_{i_0+m}p^m}$ is a homogeneous G -subbundle satisfying the hypothesis of Corollary 3.11, Corollary 3.14, Lemma 3.16 or Lemma 3.17, by Remark 3.18, we have either

$$\begin{aligned} -\deg \mathcal{W}_1 &\geq b_{i_0} + \cdots + b_{i_0+m}p^m \text{ or} \\ -\deg \mathcal{W}_1 &> n(|C_m(\mathcal{W}_1)|) + (|C_0(\mathcal{W}_1)| + \cdots + |C_{m-1}(\mathcal{W}_1)|). \end{aligned}$$

Since $-\deg \mathcal{W} \geq p^{i_0}(-\deg \mathcal{W}_1)$, the above inequalities imply that

$$\begin{aligned} -\deg \mathcal{W} &\geq b_{i_0}p^{i_0} + \cdots + b_{i_0+m}p^{i_0+m} \text{ or} \\ -\deg \mathcal{W} &> np^{i_0}|C_m(\mathcal{W}_1)| + (p^{i_0}|C_0(\mathcal{W}_1)| + \cdots + p^{i_0}|C_{m-1}(\mathcal{W}_1)|) \\ &\geq n|C_{i_0+m}(\mathcal{W})| + (|C_{i_0}(\mathcal{W})| + \cdots + |C_{i_m}(\mathcal{W})|) \\ &\geq -\mu(\mathcal{V}_d)|W|, \end{aligned}$$

where the last line follows from Remark 3.6. Hence, in both the cases, if $\mathcal{W} \subset \mathcal{V}$ then $\mu(\mathcal{W}) < \mu(\mathcal{V}_d)$.

Now, due to the uniqueness property of the Harder-Narasimhan filtration, the destabilizing subbundle \mathcal{W} of \mathcal{V}_d is a homogeneous G -subbundle such that $\mu(\mathcal{W}) > \mu(\mathcal{V}_d)$, which contradicts the result above. In particular \mathcal{V}_d is semistable. Now suppose \mathcal{V}_d is not stable then it has a subbundle $\mathcal{V}' \subset \mathcal{V}_d$ such that $\mu(\mathcal{V}') = \mu(\mathcal{V}_d)$.

Now $\text{Socle}(\mathcal{V}')$ is the unique polystable subbundle of same slope, containing \mathcal{V}' . In particular $\text{Socle}(\mathcal{V}')$ is homogeneous G -subbundle of same slope as \mathcal{V}_d . Hence $\mathcal{V}_d = \text{Socle}(\mathcal{V}')$ is polystable.

Now since $H^0(\mathbf{P}_k^n, \mathcal{E}nd(\mathcal{V}_d)) = k$, by a simple calculation we conclude that, \mathcal{V}_d must be stable.

This proves the theorem. \square

Proof of Proposition 1.3 If \mathcal{V}_d^* is semistable then

$$\mu_{\max}(\mathcal{V}_d^*) = \mu(\mathcal{V}_d^*) = \frac{d}{\binom{d+n}{d} - 1}.$$

Hence we assume that \mathcal{V}_d^* is not semistable. Let

$$0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_l = \mathcal{V}_d^*$$

be the Harder-Narasimhan filtration of \mathcal{V}_d^* . Then, by definition, $\mu_{\max}(\mathcal{V}_d^*) = \mu(\mathcal{U}_1)$. Note that \mathcal{U}_1 is a G -subbundle of \mathcal{V}_d^* , and therefore there exists corresponding P -submodule, say, U_1 of V_d . Consider the short exact sequence of G -bundles

$$0 \longrightarrow \mathcal{U}_1 \longrightarrow \mathcal{V}_d^* \longrightarrow \mathcal{E} \longrightarrow 0,$$

Taking dual of this, we get a short exact sequence of G -bundles

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{V}_d^{**} = \mathcal{V}_d \longrightarrow \mathcal{U}_1^* \longrightarrow 0.$$

Now let us denote the G -subbundle \mathcal{E}^* by \mathcal{W} and let W be the corresponding P -submodule of V_d . Now, the first inequality of the proposition follows from hypothesis that $\mu(\mathcal{V}_d^*) < \mu(\mathcal{U}_1) = \mu_{\max}(\mathcal{V}_d^*)$. Moreover

$$\mu(\mathcal{U}_1) > \mu(\mathcal{V}_d^*) \implies \mu(\mathcal{E}) < \mu(\mathcal{V}_d^*) \implies \mu(\mathcal{W}) > \mu(\mathcal{V}_d).$$

Now, by Lemma 3.7, we have $W(a_0) \neq 0$ and therefore, by Remark 3.13, we have $\deg \mathcal{W} < 0$. This gives

$$\deg \mathcal{U}_1 = \deg \mathcal{V}_d^* - \deg \mathcal{E} = \deg \mathcal{V}_d^* + \deg \mathcal{W} < d.$$

By Lemma 3.8, $W(a_0 + \cdots + a_{m-1}p^{m-1}) = 0$, as $\mu(\mathcal{W}) > \mu(\mathcal{V}_d)$. Hence, by Remark 3.2, for every $0 \leq i_m \leq a_m - 1$, we have

$$W_{a_0, \dots, a_{m-1}, i_m} = W(a_0 + \cdots + a_{m-1}p^{m-1} + i_m p^m) = 0.$$

Now

$$\begin{aligned} |U_1| &= |V_d| - |W| \\ &= \sum_{\{(i_0, \dots, i_m) | 0 \leq i_j \leq p-1, i_0 + \cdots + i_m p^m < d\}} |V_{i_0, \dots, i_m}| - |W_{i_0, \dots, i_m}| \end{aligned}$$

As $|V_{i_0, \dots, i_m}| - |W_{i_0, \dots, i_m}| \geq 0$, for every tuple (i_0, \dots, i_m) , this implies

$$\begin{aligned} |U_1| &\geq \sum_{\{i_m | 0 \leq i_m \leq a_m - 1\}} |V_{a_0, \dots, a_{m-1}, i_m}| - |W_{a_0, \dots, a_{m-1}, i_m}| \\ &\geq \sum_{0 \leq i_m \leq a_m - 1} |V_{a_0, \dots, a_{m-1}, i_m}| = |S^{a_m p^m}(V_1)| + \cdots + |S^{p^m}(V_1)| \geq |S^{\lceil d/2 \rceil}(V_1)|. \end{aligned}$$

In particular $\mu(\mathcal{U}_1) \leq d/|S^{\lceil d/2 \rceil}(V_1)|$. Hence the proposition. \square

Proof of Corollary 1.4 By the proof of Theorem 1.1 of Langer; if

$$d > \frac{r-1}{r} \Delta(E) H^{n-2} + \frac{1}{r(r-1)H^n}$$

and m the least integer such that the quotients of the Harder-Narasimhan filtration of the restriction of E to a very general divisor in dH are strongly semistable, then for general hypersurface D in $|dH|$, we have

$$\frac{d}{\max\{\frac{r^2-1}{4}, 1\}} \leq \mu_i(F^{m*}E|_D) - \mu_{i+1}(F^{m*}E|_D) \leq H^n \cdot \mu_{\max}(\mathcal{V}_d^*|_D).$$

By Proposition 1.3, this implies

$$\frac{d}{\max\{\frac{r^2-1}{4}, 1\}} \leq \mu_i(F^{m*}E|_D) - \mu_{i+1}(F^{m*}E|_D) \leq H^n \cdot \frac{d^2}{\binom{\lceil d/2 \rceil + n - 1}{\lceil d/2 \rceil}}.$$

Moreover, for $n = 2$, by Proposition 2.1, we have

$$\frac{d}{\max\{\frac{r^2-1}{4}, 1\}} \leq \mu_i(F^{m*}E|_D) - \mu_{i+1}(F^{m*}E|_D) \leq H^n \cdot \frac{d^2}{\binom{d+n}{d} - 1}.$$

Now, for d such that

(1) for $n = 2$, the inequality

$$\frac{\binom{d+n}{d} - 1}{d} > H^n \cdot \max\{\frac{r^2-1}{4}, 1\} + 1$$

holds and

(2) for $n \geq 3$, the inequality

$$\frac{\binom{\lceil d/2 \rceil + n - 1}{\lceil d/2 \rceil}}{d} > H^n \cdot \max\{\frac{r^2-1}{4}, 1\} + 1$$

holds, we have a contradiction. In particular, for d satisfying the hypothesis of the corollary, E_D is strongly semistable, for a very general $D \in |dH|$. \square

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SEMISTABILITY OF SYZYGY BUNDLES ON PROJECTIVE SPACES IN POSITIVE CHARACTERISTICS

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